TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 309, Number 1, September 1988

FUNCTION SPACES GENERATED BY BLOCKS ASSOCIATED WITH SPHERES, LIE GROUPS AND SPACES OF HOMOGENEOUS TYPE

ALEŠ ZALOŽNIK

ABSTRACT. Functions generated by blocks were introduced by M. Taibleson and G. Weiss in the setting of the one-dimensional torus T [**TW1**]. They showed that these functions formed a space "close" to the class of integrable functions for which we have almost everywhere convergence of Fourier series. Together with S. Lu [**LTW**] they extended the theory to the n-dimensional torus where this convergence result (for Bochner-Riesz means at the critical index) is valid provided we also restrict ourselves to $L \log L$. In this paper we show that this restriction is not needed if the underlying domain is a compact semisimple Lie group (or certain more general spaces of a homogeneous type). Other considerations (for example, these spaces form an interesting family of quasi-Banach spaces; they are connected with the notion of entropy) guide one in their study. We show how this point of view can be exploited in the setting of more general underlying domains.

1. Introduction. Function spaces generated by blocks were introduced by M. Taibleson and G. Weiss in the setting of the one-dimensional torus T (see [**TW1**]). The motivation for their study was the connection of these spaces with the almost everywhere convergence of Fourier series. Later S. Lu, M. Taibleson and G. Weiss extended the theory to the n-dimensional torus T^n proving almost everywhere convergence results for the Bochner-Riesz means of multiple Fourier series (see [**LTW**]).

Spaces generated by blocks have also been studied from other viewpoints. They are connected with R. Fefferman's notion of entropy (see [**F**, **SO**, **TW1**]), and with random Fourier series (see [**TW1**]), as well as being of interest on their own (see [**MTW**]). For a general overview of the theory see the survey article [**TW2**].

The purpose of this work is to prove results which motivate the study of spaces generated by blocks

(i) as classes of convergence in the setting of the unit sphere S^n and on compact semisimple Lie groups, and

(ii) as objects interesting because of their functional analytic properties in the general setting of a space of homogeneous type.

We start by a brief description of blocks and the spaces they generate on the torus T. As indicated in [**TW1**] the theory has its roots in the atomic theory of Hardy spaces; in fact, the definitions exhibit much of the flavor of the atomic

Received by the editors June 1, 1987. Parts of this article were presented by G. Weiss at the 823rd meeting of the Society in Columbia, Missouri, in November 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 43A80, 43A85.

Supported in part by the Republic of Slovenia Science Foundation.

theory. The basic building blocks are now q-blocks which are just $(1,q)$-atoms without cancellation.

A q-block, $1 < q \leq \infty$, on T is a measurable function b satisfying

(a) the localization property $\operatorname{supp} b \subset I$ for some arc $I \subset T$, and

(b) the size condition $\|b\|_q \leq |I|^{-1/q'}$, where $|I|$ is the measure of I and q' is the conjugate index of q.

The numerical sequences (m_k) used to combine the q-blocks satisfy

$$N((m_k)) = \sum_k |m_k| \left(1 + \log \frac{\sum_j |m_j|}{|m_k|}\right) < \infty$$

(see Lemma (1.1) below for motivation).

The spaces $B_q(T)$ generated by q-blocks are defined by

$$B_q(T) = \left\{f = \sum m_k b_k, N((m_k)) < \infty,\ b_k \text{ are } q\text{-blocks}\right\}$$

and are equipped with the quasi-norm

$$N_q(f) = \inf N((m_k)),$$

where the infimum is taken over all sequences (m_k) for which f has a representation $f = \sum m_k b_k$ with b_k being q-blocks. $B_q(T)$ is a quasi-Banach space.

As pointed out in [**TW2**] the basic definitions make sense in any metric space equipped with a measure suitably related to the metric; that is, B_q can be defined on any space of homogeneous type.

We now give a short description of the contents section by section.

In §§2 and 3 we study convergence problems in two special instances of a space of homogeneous type. On the unit sphere S^{n-1} of $\mathbf{R}^n$ we consider the Cesàro means of the decompositions into spherical harmonics. We establish the almost everywhere convergence of these means for functions in $B_q(S^{n-1})$. In the third section we prove analogous results for the Riesz means of Fourier series on a compact semisimple Lie group.

In §4 we generalize constructions which M. Taibleson and G. Weiss proved and used on T to establish many functional analytic properties of the spaces B_q. These constructions essentially show how the quasi-norm N_q is not only measuring the size of functions in B_q but also tells us something about the geometry of their level sets. We work in the setting of a nondiscrete space of homogeneous type.

The fifth and final section is devoted to the following lemma first proved by E. Stein and N. Weiss (we state the lemma now because we will make use of it in §§2 and 3).

(1.1) LEMMA (SEE [**TW1**]). *Let (g_k) be a sequence of measurable functions on a measure space (X, μ) such that for every $k \in \mathbf{N}$*

$$\mu(\{x \in X, |g_k(x)| > \lambda\}) \leq \lambda^{-1} \qquad (\lambda > 0)$$

and let (m_k) be a numeric sequence with $N((m_k)) < \infty$. Then

$$\mu\left(\left\{x \in X, \sum |m_k|\,|g_k(x)| > \lambda\right\}\right) \leq 4N((m_k))\lambda^{-1} \qquad (\lambda > 0),$$

in particular, the series $\sum m_k g_k$ *converges absolutely almost everywhere on* X.

This lemma is one of the basic ingredients of the theory of spaces generated by blocks. We show, using probability theory, that the condition $N((m_k)) < \infty$ is in a sense best possible for the conclusion of the lemma; this result may be of independent interest.

This work forms part of the author's dissertation completed at Washington University, St. Louis. The author wishes to express his deep gratitude to Guido Weiss for his guidance and encouragement.

2. Cesàro means on the unit sphere. In this section we study Cesàro means of expansions in spherical harmonics associated with functions defined on the unit sphere of the Euclidean space $\mathbf{R}^n$, $n \geq 3$. We show that spaces generated by blocks play the role of a class of convergence for Cesàro means of critical order much the same way as on the tori T^n.

First we fix some notation and describe the harmonic analysis on the sphere. S^{n-1} denotes the unit sphere in $\mathbf{R}^n$ and we assume that $n \geq 3$. Points on S^{n-1} are denoted by $x', y', \ldots$ and $x' \cdot y'$ is the usual inner product of x' and y' in $\mathbf{R}^n$. S^{n-1} is a Riemannian submanifold of $\mathbf{R}^n$ and the geodesic distance is given by $d(x', y') = \arccos x' \cdot y'$ $(x', y' \in S^{n-1})$ so that $0 \leq d(x', y') \leq \pi$. The Lebesgue measure on $\mathbf{R}^n$ induces a measure on S^{n-1} invariant for rotations around the origin. We denote this measure by σ and assume that σ is normalized: $\sigma(S^{n-1}) = 1$.

We have the orthogonal decomposition

$$L^2(S^{n-1}) = \bigoplus \sum_{k=0}^{\infty} H_k$$

where H_k is the space of spherical harmonics of degree k; that is, the space of restrictions to S^{n-1} of polynomials, homogeneous of degree k and harmonic on $\mathbf{R}^n$. Thus for $f \in L^2(S^{n-1})$

$$f = \sum_{k=0}^{\infty} f_k \qquad (f_k \in H_k),$$

the series converging in $L^2(S^{n-1})$.

The orthogonal projections $f \to f_k$ are given by

$$f_k(x') = \langle f, Z_{x'}^{(k)} \rangle = \int_{S^{n-1}} f(y') Z_{x'}^{(k)}(y')\, d\sigma(y'), \tag{2.1}$$

$Z_{x'}^{(k)}$ being the zonal harmonic of degree k with pole x'; that is, the unique element of H_k invariant under rotations leaving x' fixed and normalized so that $Z_{x'}^{(k)}(x') = \dim H_k$. Note that although we assume functions to be complex valued, the zonal harmonics are real valued (see [**SW**, p. 143]).

The integral in (2.1) makes sense for every integrable function f; therefore, we assign to each $f \in L^1(S^{n-1})$ its decomposition:

$$f \to \sum_{k=0}^{\infty} f_k \qquad (f \in L^1(S^{n-1}),\ f_k \in H_k), \tag{2.2}$$

where the f_k are given by (2.1).

We are interested in convergence properties of this series; in particular in its Cesàro summability.

For $\delta > -1$ the Cesàro means of order δ of the series (2.2) are defined by

$$\sigma_L^\delta f = (A_L^\delta)^{-1} \cdot \sum_{k=0}^{k=L} A_{L-k}^\delta \cdot f_k, \qquad (L = 0, 1, 2, \dots)$$

where the coefficients A_L^δ are given by

$$A_0^\delta = 1, \quad A_L^\delta = \binom{L+\delta}{L} = \frac{(\delta+L)\cdots(\delta+2)(\delta+1)}{L\cdots 2\cdot 1} \qquad (L = 1, 2, \dots).$$

σ_L^δ is an integral operator whose kernel is a finite weighted sum of zonal harmonics.

The invariance of $Z_{x'}^{(k)}$ under rotations leaving x' fixed implies

$$Z_{x'}^{(k)}(y') = p_k(x' \cdot y') \qquad (k = 0, 1, 2, \dots) \tag{2.3}$$

where p_k is a real-valued polynomial of degree k on the interval $I = [-1, 1]$. The polynomials p_k are constant multiples of Gegenbauer polynomials (see [**SW**, p. 149]).

Rewriting the orthogonality relations among $Z_{x'}^{(k)}$'s in terms of Gegenbauer polynomials, one gets

$$\int_{-1}^{1} p_k(u) \cdot p_l(u) \cdot (1-u^2)^{(n-3)/2}\, du = C \cdot p_k(1) \cdot \delta_{kl}; \tag{2.4}$$

that is, $\{p_k\}_{k=0}^\infty$ is an orthogonal system of polynomials in $L^2(I, (1-u^2)^{(n-3)/2}\, du)$. Here the constant C depends only on the dimension n.

Let us make a general observation about Cesàro means. Let μ be a positive Borel measure on $I = [-1, 1]$ and let $\{q_k\}_{k=0}^\infty$ be an orthogonal system in $L^2(I, d\mu)$. The Cesàro means of order $\delta > -1$ of the expansion of a function $f \in L^2(I, d\mu)$ with respect to $\{q_k\}_{k=0}^\infty$ are

$$\sigma_L^\delta g(w) = \int_I g(u) \cdot S_L^\delta(u, w)\, d\mu(u) \qquad (w \in I,\ L = 0, 1, 2, \dots),$$

where

$$S_L^\delta(u, w) = (A_L^\delta)^{-1} \cdot \sum_{k=0}^{k=L} A_{L-k}^\delta \cdot \|q_k\|^{-2} \cdot q_k(w) \cdot \overline{q_k(u)} \tag{2.5}$$

and $\|\ \|$ is the usual norm in $L^2(I, d\mu)$.

Now, if the q_k are polynomials and $\deg q_k = k$ then the q_k's are uniquely determined up to a constant factor by the measure μ. A glance at (2.5) reveals that the constant factor cancels out for S_L^δ. Therefore the kernel function S_L^δ does not depend on how we normalize the polynomials q_k.

Let us return to S^{n-1}. We have, using (2.3), (2.4), $\deg p_k = k$ and the observation above,

$$\sigma_L^\delta f(x') = C \int_{S^{n-1}} f(y') S_L^\delta(x' \cdot y', 1)\, d\sigma(y') \qquad (f \in L^2(S^{n-1})),$$

where S_L^δ is the Cesàro means kernel for $L^2(I, (1-u^2)^{(n-3)/2}\, du)$ with respect to $\{p_k\}_{k=0}^\infty$ (see (2.5)).

To sum up: The Cesàro means kernel for the expansion in spherical harmonics is intimately related to the Cesàro means kernel of $L^2(I,(1-u^2)^{(n-3)/2}\,du)$ with respect to $\{p_k\}_{k=0}^{\infty}$. Actually, one needs only the values $S_L^{\delta}(u,1)$ $(u \in I)$ and to estimate these values one can use any orthogonal system of polynomials with increasing degrees.

Using the system $\{p_k^{(n-3)/2,(n-3)/2}\}_{k=0}^{\infty}$ of Jacobi polynomials and Szegö's treatise, A. Bonami and J. L. Clerc proved the following inequalities:

(2.6)

$$|S_L^{\delta}(\cos\theta,1)| \le C \begin{cases} (L+1)^{n/2-1-\delta}[(L+1)^{-1}+\theta]^{-n/2-\delta}, & 0 \le \theta \le \pi/2, \\ (L+1)^{n/2-1-\delta}[(L+1)^{-1}+\pi-\theta]^{-n/2+1}, & \pi/2 \le \theta \le \pi \end{cases}$$

(see [**BC**, p. 234]; since their estimates are more precise for θ close to 0 and close to π, but we need them only in the above simplified form; compare also [**CTW**, Theorem 3.21]).

As is well known, when studying a.e. convergence of Cesàro means, the maximal Cesàro means operator on $L^1(S^{n-1})$, defined by

$$\mathscr{C}^{\delta} f(x') = \sup\{|\sigma_L^{\delta} f(x')|, L = 0,1,2,\dots\},$$

plays a very important role.

Let us now state the known convergence results. A. Bonami and J. L. Clerc showed:

(2.7) If $\delta > (n-2)/2$ then $\mathscr{C}^{\delta}$ is of weak type $(1,1)$; that is,

$$\sigma(\{x' \in S^{n-1}, \mathscr{C}^{\delta} f(x') > \lambda\}) \le C \cdot \|f\|_1 \cdot \lambda^{-1} \qquad (\lambda > 0)$$

(see [**BC**, Theorem 3.3, p. 237]).

(2.8) If $1 < q \le 2$ and $\delta > (n-2)(1/q - 1/2)$ then the operator $\mathscr{C}^{\delta}$ is of strong type (q,q); that is,

$$\|\mathscr{C}^{\delta} f\|_q \le C_q \cdot \|f\|_q \qquad (f \in L^q(S^{n-1}))$$

and $\sigma_L^{\delta} f(x') \to f(x')$ a.e. on S^{n-1} as $L \to \infty$ for every $f \in L^q(S^{n-1})$ (see [**BC**, Corollary 3.4, p. 237]).

REMARK. A. Bonami and J. L. Clerc state only that $\sigma_L^{\delta} f(x')$ converges a.e. for every $f \in L^q(S^{n-1})$. If f is a finite linear combination of spherical harmonics, $\sigma_L^{\delta} f$ clearly converges to f a.e. Since $\mathscr{C}^{\delta} f$ is in $L^q(S^{n-1})$ we also have $\|\sigma_L^{\delta} f - f\|_q \to 0$ as $L \to \infty$ by the dominated convergence theorem. Finite linear combinations of spherical harmonics form a dense subspace of $L^q(S^{n-1})$, therefore $\|\sigma_L^{\delta} f - f\|_q \to 0$ as $L \to \infty$ for every $f \in L^q(S^{n-1})$ so the a.e. limit of $\sigma_L^{\delta} f$ must be f itself.

The next result was announced by E. Stein, M. Taibleson and G. Weiss (see [**STW**, p. 96]) and proved by L. Colzani, M. Taibleson and G. Weiss (see [**CTW**, Theorem 4.1, p. 881]).

(2.9) If $0 < p < 1$ and $\delta = (n-1)/p - n/2$ then the operator $\mathscr{C}^{\delta}$ is well defined on the atomic space $H^p(S^{n-1})$ and maps it boundedly into weak-$L^p(S^{n-1})$; that is,

$$\sigma(\{x \in S^{n-1}, \mathscr{C}^{\delta} f(x') > \lambda\}) \le C \cdot \|f\|_{H^p} \cdot \lambda^{-p} \qquad (\lambda > 0).$$

If $p = 1$ and $\delta = (n-2)/2$ then (2.9) fails; this was proven by M. Taibleson (see [**T**] and the remarks in [**CTW**, p. 883]).

We shall define function spaces generated by blocks on S^{n-1} and then show that they can serve as a substitute for atomic $H^1(S^{n-1})$ at the critical index $\delta = (n-2)/2$.

For $x_0' \in S^{n-1}$ and $0 < \theta_0 \leq \pi$ a "cap" of radius θ_0 centered at x_0' is the set

$$B(x_0', \theta_0) = \{x' \in S^{n-1}, d(x', x_0') < \theta_0\}.$$

For $1 < q \leq \infty$ a q-block on S^{n-1} is a measurable function b defined on S^{n-1} and such that

$$\text{(2.10)} \qquad \operatorname{supp} b \subset B = B(x_0', \theta_0) \quad \text{for some } x_0' \in S^{n-1},\ \theta_0 > 0$$

and

$$\text{(2.11)} \qquad \left(\int_B |b(x')|^q \frac{d\sigma(x')}{\sigma(B)}\right)^{1/q} \leq \frac{1}{\sigma(B)}.$$

Normalization (2.11) and Hölder's inequality imply $\|b\|_1 \leq 1$. It is also clear, by Hölder's inequality again, that if $q_2 > q_1$, then every q_2-block is also q_1-block. It is often useful to have (2.11) in the following form:

$$\text{(2.12)} \qquad \|b\|_q \leq (\sigma(B))^{(1-q)/q} = C \cdot \theta_0^{(n-1)(1-q)/q},$$

where C depends on q and n.

The main result of this section is a uniform weak type $(1,1)$ inequality for the action of $\mathscr{C}^{(n-2)/2}$ on q-blocks.

(2.13) THEOREM. *If $1 < q \leq \infty$ and if b is a q-block, then*

$$\sigma(\{x' \in S^{n-1}, \mathscr{C}^{(n-2)/2} b(x') > \lambda\}) \leq C\lambda^{-1} \qquad (\lambda > 0),$$

where the constant C does not depend on b and λ.

PROOF. We prove the estimate for large λ's first: Let us assume that $\lambda > \theta_0^{1-n}$ where θ_0 is the radius of the cap on which b is supported.

If $1 < q \leq 2$ then, since $(n-2)/2 > (n-2)(1/q - 1/2)$, we may use (2.8) which says that our operator is of strong type (q,q) and therefore also of weak type (q,q). By the normalization condition (2.12) we have for all $\lambda > \theta_0^{1-n}$

$$\sigma(\{x' \in S^{n-1}, \mathscr{C}^{(n-2)/2} b(x') > \lambda\}) \leq C \cdot \frac{\|b\|_q^q}{\lambda^q} \leq C \cdot \frac{\theta_0^{(1-n)(q-1)}}{\lambda^q} < C\lambda^{-1}.$$

For $2 < q \leq \infty$ we get the same estimate simply because every q-block, $q > 2$, is also a 2-block.

Now let us consider the case $\lambda \leq \theta_0^{1-n}$ and let

$$A_1 = \{x' \in S^{n-1}, 2 \cdot \theta_0 \leq d(x_0', x') \leq \pi/2\},$$
$$A_2 = \{x' \in S^{n-1}, \pi/2 \leq d(x_0', x') \leq \pi - 2 \cdot \theta_0\},$$

where x_0' is the center of the cap $B = B(x_0', \theta_0)$ containing the support of b. The set $S^{n-1} \backslash (A_1 \cup A_2)$ has measure of order λ^{-1}, so if $A_1 \cup A_2$ is empty, which happens if b has large support, there is nothing to prove. If $A_1 \cup A_2$ is not empty, we will show, using (2.6), that the subset of $A_1 \cup A_2$ where $\mathscr{C}^{(n-2)/2} b(x') > \lambda$ also has measure of the order λ^{-1}.

If $y' \in B$ and $x' \in A_1$ then

$$\begin{aligned} d(x_0', x') &\le d(x_0', y') + d(y', x') \\ &\le \theta_0 + d(y', x') \le \tfrac{1}{2} \cdot d(x_0', x') + d(y', x') \end{aligned}$$

and

$$\begin{aligned} d(x_0', x') &\le d(-x_0', x') \le d(-x_0', -y') + d(-y', x') \\ &\le \theta_0 + d(-y', x') < \tfrac{1}{2} \cdot d(x_0', x') + \pi - d(x', y'). \end{aligned}$$

Therefore,

$$d(x_0', x') \le 2 \cdot \min\{d(x', y'), \pi - d(x', y')\}.$$

According as $d(x', y') \le \pi/2$ or $> \pi/2$ (the latter may happen!) we now use the first or the second estimate (2.6) and get, using also $-n/2 - \delta = -n+1 < -n/2+1$,

$$|S_L^{(n-2)/2}(x' \cdot y', 1)| \le C \cdot (d(x_0', x'))^{1-n} \qquad (x' \in A_1, y' \in \operatorname{supp} b).$$

Similarly, if $y' \in B$ and $x' \in A_2$, then

$$\begin{aligned} d(-x_0', x') &\le d(x_0', x') < \theta_0 + d(y', x') \\ &\le \tfrac{1}{2}(\pi - d(x_0', x')) + d(y', x') = \tfrac{1}{2} d(-x_0', x') + d(y', x') \end{aligned}$$

and

$$\begin{aligned} d(-x_0', x') &\le d(-x_0', -y') + d(-y', x') \\ &< \theta_0 + d(-y', x') \le \tfrac{1}{2} d(-x_0', x') + \pi - d(y', x'), \end{aligned}$$

so that,

$$d(-x_0', x') \le 2 \cdot \min\{d(x', y'), \pi - d(x', y')\}.$$

Using (2.6) as before we get the estimate

$$|S_L^{(n-2)/2}(x' \cdot y', 1)| \le C \cdot (\pi - d(x_0', x'))^{1-n} \qquad (x' \in A_2, y' \in \operatorname{supp} b).$$

Now, since

$$\mathscr{C}^{(n-2)/2} b(x') \le \|b\|_1 \cdot \sup\{|S_L^{(n-2)/2}(x' \cdot y', 1)|, y' \in \operatorname{supp} b, L = 0, 1, 2, \dots\}$$

and $\|b\|_1 \le 1$, we finally get

$$\begin{aligned} \sigma(\{x' \in S^{n-1}, &\mathscr{C}^{(n-2)/2} b(x') > \lambda\}) \le \sigma(B(x_0', 2\theta_0)) + \sigma(B(-x_0', 2\theta_0)) \\ &+ \sigma(\{x' \in A_1, \mathscr{C}^{(n-2)/2} b(x') > \lambda\}) + \sigma(\{x' \in A_2, \mathscr{C}^{(n-2)/2} b(x') > \lambda\}) \\ &\le \sigma(B(x_0', 2\theta_0)) + \sigma(B(-x_0', 2\theta_0)) \\ &\quad + \sigma(\{x' \in A_1, C(d(x_0', x'))^{1-n} > \lambda\}) + \sigma(\{x' \in A_2, C(d(x_0', x'))^{1-n} > \lambda\}) \\ &\le C \cdot \left(\theta_0^{n-1} + \frac{1}{\lambda} + \frac{1}{\lambda}\right) \le \frac{C}{\lambda} \end{aligned}$$

for every $\lambda < \theta_0^{1-n}$ and this finishes the proof. □

The spaces $B_q(S^{n-1})$, $1 < q \le \infty$, of functions generated by blocks are defined as usual. $B_q(S^{n-1})$, or simply B_q, consists of those measurable functions f which admit a representation

$$f = \sum_{k=0}^{\infty} m_k b_k \tag{2.14}$$

where the b_k's are q-blocks and (m_k) is a numeric sequence satisfying $N((m_k)) < \infty$. By Lemma (1.1) the series (2.14) converges a.e. and in L^1. The infimum of the values $N((m_k))$, taken over all representations of f as in (2.14), is a quasi-norm on B_q; we denote it by $N_q(f)$. B_q is a quasi-Banach space.

Now we prove our main convergence result.

(2.15) THEOREM. *Let $1 < q \leq \infty$. The operator $\mathscr{C}^{(n-2)/2}$ maps B_q boundedly into weak-L^1; that is, for all $f \in B_q$ and all $\lambda > 0$*

$$\sigma(\{x' \in S^{n-1}, \mathscr{C}^{(n-2)/2} f(x') > \lambda\}) \leq \frac{4 \cdot C \cdot N_q(f)}{\lambda}. \tag{2.16}$$

Here C is the constant of Theorem (2.14), independent of f and λ. As a consequence $\sigma_L^{(n-2)/2} f \to f$ a.e. on S^{n-1} as $L \to \infty$ for every $f \in B_q$.

PROOF. Let $f \in B_q$ and take any representation (2.14) of f by q-blocks. By Theorem (2.13)

$$\sigma(\{x' \in S^{n-1}, \mathscr{C}^{(n-2)/2} b_k(x') > \lambda\}) \leq \frac{C}{\lambda}, \qquad \lambda > 0,$$

for $k = 0, 1, 2, \ldots$. Therefore, by Lemma (1.1)

$$\sigma\left(\left\{x' \in S^{n-1}, \sum_{k=0}^{\infty} |m_k| \cdot \mathscr{C}^{(n-2)/2} b_k(x') > \lambda\right\}\right) \leq \frac{4 \cdot C \cdot N_q((m_k))}{\lambda} \tag{2.17}$$

for every $\lambda > 0$.

It is easy to see that

$$\mathscr{C}^{(n-2)/2} f(x') \leq \sum_{k=0}^{\infty} |m_k| \mathscr{C}^{(n-2)/2} b_k(x'). \tag{2.18}$$

Indeed, the series (2.14) converges in L^1 and $S_L^{(n-2)/2}(\cdot, 1)$, being a polynomial, is bounded, therefore

$$\begin{aligned} &|\sigma_L^{(n-2)/2} f(x') - \sum_{k=0}^{K} m_k \cdot \sigma_L^{(n-2)/2} b_k(x')| \\ &\qquad \leq \sup\{|S_L^{(n-2)/2}(x' \cdot y', 1)|, y' \in S^{n-1}\} \cdot \left\| f - \sum_{k=0}^{K} m_k b_k \right\|_1 \to 0 \end{aligned}$$

as $K \to \infty$; we get (2.18) by taking absolute values and the supremum over L in the equation

$$\sigma_L^{(n-2)/2} f = \sum_{k=0}^{\infty} m_k \cdot \sigma_L^{(n-2)/2} b_k.$$

Now (2.17), (2.18) and the fact that the representation of f by q-blocks was arbitrary imply (2.16).

To prove that the Cesàro means of f converge to f a.e., we use a standard argument. Let for $j = 1, 2, \ldots$

$$A_j = \left\{x' \in S^{n-1}, \limsup_{L \to \infty} |\sigma_L^{(n-2)/2} f(x') - f(x')| > \frac{1}{j}\right\}.$$

We will show that $\sigma(A_j) = 0$ $(j = 1, 2, \dots)$ therefore $\sigma(\bigcup A_j) = 0$ and this clearly implies the desired conclusion.

Let $\varepsilon > 0$ be arbitrary, choose a representation (2.14) of f and write

$$f = \sum_{k=0}^{K} m_k b_k + \sum_{k=K+1}^{\infty} m_k b_k = g + r,$$

where K is chosen so that

$$N((m_k)_{k=K+1}^{\infty}) < \delta = \frac{\varepsilon}{4(4C+1)j}.$$

Note that this choice implies

$$\|r\|_1 \leq N_q(r) < \delta.$$

The function g is a finite linear combination of q-blocks; therefore, it belongs to any $L^p(S^{n-1})$, $p \leq q$. So by (2.8) the Cesàro means of order $(n-2)/2$ of g converge to g a.e. on S^{n-1}. Combining this with (2.16) applied to the remainder r we get

$$\begin{aligned}
\sigma(A_j) &\leq \sigma\left(\left\{x' \in S^{n-1}, \limsup_{L\to\infty} |\sigma_L^{(n-2)/2} r(x') - r(x')| > \frac{1}{j}\right\}\right) \\
&\leq \sigma\left(\left\{x' \in S^{n-1}, \mathscr{C}^{(n-2)/2} r(x') > \frac{1}{4\cdot j}\right\}\right) \\
&\quad + \sigma\left(\left\{x' \in S^{n-1}, |r(x')| > \frac{1}{4\cdot j}\right\}\right) \\
&< 4 \cdot C \cdot \delta \cdot 4j + \delta \cdot 4j = \varepsilon.
\end{aligned}$$

Since ε was arbitrary, $\sigma(A_j) = 0$, and the theorem is proved. □

3. Riesz means on compact semisimple Lie groups. In this section we show that spaces generated by blocks may be of interest as classes of convergence also in the context of compact Lie groups; in particular, we study the behaviour of the Riesz means of Fourier series of blocks and prove convergence results analogous to those for the unit sphere in the previous section.

Let G be a compact topological group equipped with the left and right invariant Haar measure σ, $\sigma(G) = 1$. By the Peter-Weyl theorem we have the orthogonal decomposition

$$L^2(G) = \bigoplus \sum_{\lambda\in\Lambda} W_\lambda$$

where W_λ are the nontrivial subspaces of $L^2(G)$ invariant under both the right and left regular representation of G on $L^2(G)$ and Λ is the set of equivalence classes of irreducible unitary representations of G. The projection P_λ of $L^2(G)$ onto W_λ is given by $f \to d_\lambda \cdot (\chi_\lambda * f)$ where d_λ and χ_λ are the dimension and the character of a representation of class λ and

$$(\chi_\lambda * f)(x) = \int_G \chi_\lambda(y^{-1}x) f(y)\, d\sigma(y).$$

Thus we have the Fourier expansion

$$f = \sum_{\lambda\in\Lambda} P_\lambda f = \sum_{\lambda\in\Lambda} d_\lambda \cdot (\chi_\lambda * f) \qquad (f \in L^2(G)), \tag{3.1}$$

the series converging in $L^2(G)$. Since χ_λ is bounded, we can form such a series for any integrable f.

To study convergence and summability properties of such series one, first, has to decide how to order the terms in the series and, then, estimate the dimensions and the characters. To accomplish this, more structure on G is required; from now on we shall assume that G is a compact connected Lie group. For motivation let us consider the abelian case briefly; that is, let G be the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ for a moment. The characters are

$$\chi_m([x]) = \exp(2\pi i m \cdot x) \qquad ([x] \in T^n, x \in \mathbb{R}^n, m \in \mathbb{Z}^n).$$

Now, the Peter-Weyl decomposition is a simultaneous eigenspace decomposition of $L^2(G)$ for all linear operators on $L^2(G)$ that commute with translations. A very special operator of this kind is the Laplacian Δ and the characters χ_m form a complete system of solutions of the eigenvalue problem $\Delta u = -c \cdot u$. The eigenvalue of the eigenfunction χ_m is $4\pi^2|m|^2$, and to quote S. Bochner (see [**B1**, p. 179]):

> Writing the series $\sum_{m\in\mathbb{Z}^n}(\chi_m * f)$ in the form
>
> $$\sum_{j=0}^{\infty} \sum_{|m|^2=j} (\chi_m * f)$$
>
> satisfies the very natural principle of ordering the terms in the series according to the magnitude of the eigenvalues of the Laplacian.

The structure theory of compact connected Lie groups, which we describe briefly (see [**BD**] for details), allows us to apply the same natural principle in more general settings. Moreover, the culmination of the structure theory, the Weyl character and dimension formulae enable the analyst to obtain the estimates for the characters and dimensions.

Let G be a compact connected Lie group of dimension n and T a fixed maximal torus in G, $\dim T = \operatorname{rank} G = k$. Let W be the Weyl group of G, that is $W = N/T$, where N is the normalizer of T in G. Central functions on G correspond exactly to functions on T invariant under the action of W. In particular, the restriction of a character of G to T is a W-invariant character of T and, more important, any W-invariant character of T can be extended to a central function on G, which turns out to be a character of G. A representation of G is determined up to equivalence by the restriction of its character to T.

Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T respectively. The adjoint representation $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ describes the infinitesimal structure of conjugation in G and its restriction to T plays the key role when one studies the W-invariant functions on T. We can always choose an Ad-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, so that the extension of Ad to the complexification of $\mathfrak{g}$ becomes unitary. R will denote the set of all roots of G; that is the set of all $\alpha \in \mathfrak{t}^*\backslash\{0\}$ for which

$$\{X \in \mathfrak{g},\ [H, X] = 2\pi i\alpha(H) \cdot X \text{ for every } H \in \mathfrak{t}\} \neq \{0\}.$$

The action of W on T induces the action of W on $\mathfrak{t}$:

$$w \cdot H = \mathrm{Ad}(n) \cdot H \quad \text{if } w = nT \qquad (n \in N,\ H \in \mathfrak{t})$$

and so the restriction of the inner product $\langle\cdot,\cdot\rangle$ to $\mathfrak{t}$ is W-invariant. We use the natural isomorphism $\mathfrak{t}^* \to \mathfrak{t}$ given by $\lambda \to H_\lambda$ where $\lambda(H) = \langle H_\lambda, H\rangle$ $(H \in \mathfrak{t})$ to transfer $\langle\cdot,\cdot\rangle$ and the action of W to $\mathfrak{t}^*$:

$$\langle \lambda, \iota\rangle = \langle H_\lambda, H_\iota\rangle \qquad (\lambda, \iota \in \mathfrak{t}^*),$$
$$H_{w\cdot\lambda} = w \cdot H_\lambda \qquad (\lambda \in \mathfrak{t}^*,\ w \in W),$$

that is, $(w\cdot\lambda)(H) = \langle w\cdot H_\lambda, H\rangle = \langle H_\lambda, w\cdot H\rangle = \lambda(w\cdot H)$ $(H \in \mathfrak{t})$.

The kernel of the exponential map $\mathfrak{t} \to T$, that is, the integral lattice is denoted by I and

$$I^* = \{\alpha \in \mathfrak{t}^*, \alpha(I) \subset \mathbb{Z}\}$$

is the lattice of integral forms. We have $R \subset I^*$. We fix a Weyl chamber $\mathfrak{t}^+$, that is, a connected component of $\mathfrak{t}\setminus \bigcup_{\alpha\in R} \ker\alpha$ and define the associated positive roots by

$$R_+ = \{\alpha \in R, \alpha(H) > 0 \text{ for every } H \in \mathfrak{t}^+\}.$$

The half-sum of the positive roots is denoted by ρ. For $\lambda \in \mathfrak{t}^*$ we define the alternating sum of λ by

$$A(\lambda)\colon \mathfrak{t} \to \mathbb{C},$$
$$A(\lambda)(H) = \sum_{w\in W} \det(w)\cdot \exp(2\pi i\lambda(w\cdot H)) \qquad (H \in \mathfrak{t}),$$

where $\det(w)$ is the determinant of w as an element of $O(\mathfrak{t})$, so that $\det(w) = 1$ or -1. The Weyl denominator is the function

$$D\colon \mathfrak{t} \to \mathbb{C},$$
$$D(H) = \prod_{\alpha\in R_+} [\exp(\pi i\alpha(H)) - \exp(-\pi i\alpha(H)]$$
$$= \prod_{\alpha\in R_+} 2i\cdot \sin \pi\alpha(H) \qquad (H \in \mathfrak{t}).$$

We list some facts about this function. D is the alternating sum of the form ρ: $D = A(\rho)$. The function $|D|^2$ factors through $\exp\colon \mathfrak{t} \to T$; that is, a function $E\colon T \to \mathbb{C}$ can be well defined by $|D|^2 = E\circ\exp$. Moreover, E appears in the Weyl integral formula

$$\int_G f(x)\, d\sigma(x) = (\operatorname{card} W)^{-1}\cdot \int_T f(t)\cdot E(t)\, d\omega(t)$$

valid for central functions f on G. Here ω is the normalized Haar measure on T. The following facts are due to H. Weyl.

(i) If $\lambda \in I^* \subset \mathfrak{t}^*$, then the function $c(\lambda+\rho) = A(\lambda+\rho)/D$, first defined only when $D(H) \neq 0$, has a unique continuous extension to all of $\mathfrak{t}$ and the function $C(\lambda+\rho)\colon T \to \mathbb{C}$ in $c(\lambda+\rho) = C(\lambda+\rho)\circ\exp$ is well defined.

(ii) If $\lambda \in I^* \cap \overline{\mathfrak{t}^+}$, then $C(\lambda+\rho)$ is the restriction to T of an irreducible character χ_λ of G. The dimension of this representation is

$$d_\lambda = \prod_{\alpha\in R_+} \frac{\langle\alpha, \lambda+\rho\rangle}{\langle\alpha,\rho\rangle}.$$

(iii) The map $\lambda \to C(\lambda + \rho)$ sets up a bijective correspondence between $I^* \cap \overline{\mathfrak{t}^+}$ and the set of all irreducible characters of G; therefore, we may identify $I^* \cap \overline{\mathfrak{t}^+}$ and Λ as sets.

Now we relate Λ to a Laplacian on G. Let $X_1, X_2, \ldots, X_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to our chosen Ad-invariant inner product. We form the second order left invariant differential operator

$$\Delta = \sum_{i=1}^{n} X_i^2,$$

which turns out to be also right invariant. Δ does not depend on the choice of the basis $\{X_i\}$ and we call it the Laplace operator on G (Δ is the Laplace-Beltrami operator for the Riemannian structure on G induced by $\langle \cdot, \cdot \rangle$). If χ_λ is the character of the irreducible representation corresponding to $\lambda \in I^* \cap \overline{\mathfrak{t}^+}$ then

$$\Delta \chi_\lambda = -(\langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle) \cdot \chi_\lambda = -\mu_\lambda \cdot \chi_\lambda.$$

Since $I^* \cap \overline{\mathfrak{t}^+}$ is a part of a finite-dimensional lattice the values μ_λ ($\lambda \in I^* \cap \overline{\mathfrak{t}^+}$) form a discrete set in $\mathbb{R}^+$. Therefore, by ordering the terms of the series (3.1) with increasing values μ_λ we shall satisfy the natural principle, mentioned before, of ordering terms according to the magnitude of eigenvalues of a Laplacian (see also [**B2**]).

REMARK. The differential operator Δ depends on the choice of the inner product on $\mathfrak{g}$ so is far from being unique. If $\mathfrak{g}$ is semisimple, the negative of the Killing form may be considered as a canonical choice of $\langle \cdot, \cdot \rangle$. In general, since G is compact, $\mathfrak{g}$ is reductive, that is $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}_1$ is semisimple. Taking minus the Killing form on $\mathfrak{g}_1$, any inner product on $\mathfrak{z}$ and requiring $\mathfrak{z} \perp \mathfrak{g}_1$ yields an almost canonical Ad-invariant inner product on $\mathfrak{g}$.

A general method of summation of the series (3.1) can now be described as follows. Let Φ be a function in $C_0(\mathbb{R}^+)$ satisfying $\Phi(0) = 1$. The Φ-means of the series (3.1) are

$$\sum_{\lambda \in \Lambda} \Phi(\varepsilon \cdot \mu_\lambda) \cdot d_\lambda \cdot (\chi_\lambda * f) \qquad (\varepsilon > 0)$$

and one asks, when and in what sense do the Φ-means of the series associated to f converge to f as $\varepsilon \to 0$. We will be interested in the case when

$$\Phi(t) = (1-t)_+^\delta = \max\{(1-t)^\delta, 0\} \qquad (t \geq 0,\ \delta > 0).$$

This choice gives the Riesz means of order $\delta > 0$:

$$\rho_\varepsilon^\delta = S_\varepsilon^\delta * f \qquad (f \in L^2(G),\ \varepsilon > 0)$$

where

$$S_\varepsilon^\delta = \sum_{\lambda \in \Lambda} (1 - \varepsilon \cdot \mu_\lambda)_+^\delta \cdot d_\lambda \cdot \chi_\lambda.$$

When G is simply connected (and thus semisimple) it possesses a fundamental system of weights; that is, a set $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ of weights ($k = \operatorname{rank} G$) such that $\Lambda = I^* \cap \overline{\mathfrak{t}^+}$ is the set of all linear combinations of $\lambda_1, \lambda_2, \cdots, \lambda_k$ with nonnegative integer coefficients. This nice structure of Λ (and Weyl's formulae, of course) allowed J. L. Clerc (see [**C**]) to reduce the problem of estimating S_ε^δ to the problem of estimating the first derivatives of the corresponding kernel in the abelian case.

He obtained:

(3.2) Let G be simply connected with rank k and dimension n. If $\delta > (k-1)/2$ then

$$|S_\varepsilon^\delta(\exp H)| \le C \cdot \varepsilon^{\delta-(n-1)/2}[\|H\|^{-(n+1)/2-\delta} + |D(H)|^{-1}] \tag{3.3}$$

for every $H \in \mathfrak{t}$, where $\|H\| = (\langle H, H\rangle)^{1/2}$ (see [**C**, Theorem 2]).

Using this estimate, J. L. Clerc proved the following convergence results:

(3.4) If $\delta > (n-1)/2$ and $1 \le p < \infty$, then $\rho_\varepsilon^\delta f$ converges to f in $L^p(G)$ as $\varepsilon \to 0$ for every $f \in L^p(G)$ (see [**C**, Theorem 3]).

(3.5) If $\delta > (n-1)/2$ then the maximal operator

$$\mathscr{R}^\delta f(x) = \sup_{\varepsilon>0} |\rho_\varepsilon^\delta f(x)| \qquad (f \in L^1(G),\ x \in G)$$

is of weak type $(1,1)$; that is,

$$\sigma(\{x \in G, \mathscr{R}^\delta f(x) > \eta\}) \le C \cdot \|f\|_1 \cdot \eta^{-1} \qquad (\eta > 0)$$

and $\rho_\varepsilon^\delta f(x) \to f(x)$ a.e. on G as $\varepsilon \to 0$ for every $f \in L^1(G)$ (see [**C**, Theorem 5]).

The result (3.4) for the special case $p = 2$ can be extended to the whole range $\delta > 0$ by an argument atributed to Kaczmarcz (see [**SW**, VII.5.10] for this argument in the case $G = T^n$). Interpolating between this fact and the weak $(1,1)$ result J. L. Clerc obtained convergence below the critical index $(n-1)/2$:

(3.6) If $1 < p \le 2$ and $\delta > (n-1)(1/p - 1/2)$ then $\mathscr{R}^\delta$ is of strong type (p,p) and $\rho_\varepsilon^\delta f(x) \to f(x)$ a.e. on G as $\varepsilon \to 0$ for every $f \in L^p(G)$ (see [**C**, Corollary to Theorem 5]).

We shall be interested in the situation for the critical index $\delta = (n-1)/2$ and we assume that G is simply connected. Then $n > k$ so we may use (3.2). Note that the right-hand side in (3.2) does not depend on ε if $\delta = (n-1)/2$. We get

$$\sup_{\varepsilon>0} |S_\varepsilon^{(n-1)/2}(\exp H)| \le A \cdot (\|H\|^{-n} + D(H)|^{-1}). \tag{3.7}$$

Also since $(n-1)/2 > (n-1)(1/p - 1/2)$ if $1 < p \le 2$, (3.6), in particular, states

$$\|\mathscr{R}^{(n-1)/2} f\|_p \le C_p \cdot \|f\|_p \qquad (f \in L^p(G),\ 1 < p \le 2). \tag{3.8}$$

Now we define blocks on G and show, using (3.7) and (3.8), that the space generated by blocks is a class of convergence for the Riesz means of critical order $(n-1)/2$.

By $d(\cdot,\cdot)$ we denote the Riemannian metric on G induced by the Ad-invariant inner product on $\mathfrak{g}$ (which is minus the Killing form, since G is semisimple). If $1 < p \le \infty$ then a measurable function b defined on G is a p-block if

$$\operatorname{supp} b \subset B(x,\eta) = \{y \in G,\ d(x,y) < \eta\}$$

and

$$\|b\|_p \le (\sigma(B(x,\eta)))^{(1-p)/p}.$$

Hölder's inequality implies that a q-block is a p-block if $q > p$ and that $\|b\|_1 \le 1$ for every p-block, $1 < p \le \infty$. Note also that

$$\sigma(B(x,\eta)) \le C\eta^n,$$

because the Haar measure is also the Riemannian measure.

(3.9) THEOREM. *If b is a p-block, $1 < p \leq \infty$, then*

$$\sigma(\{x \in G, \mathscr{R}^{(n-1)/2} b(x) > \lambda\}) \leq C\lambda^{-1} \qquad (\lambda > 0)$$

where the constant C does not depend on b and λ.

PROOF. We may assume that b is supported in a ball $B = B(e, \eta)$. Consider first the case when $\lambda \geq \sigma(B)^{-1}$. Then by the normalization condition for blocks $\|b\|_p^p \leq \lambda^{p-1}$. For $1 < p \leq 2$ we use (3.8) to deduce

$$\sigma(\{x \in G, \mathscr{R}^{(n-1)/2} b(x) < \lambda\}) \leq C \cdot \|b\|_p^p \cdot \lambda^{-p} \leq C\lambda^{-1}.$$

For $p > 2$ we simply use the fact that a p-block is then also a 2-block.

Assume now that $\lambda < \sigma(B)^{-1}$.

We first rewrite estimate (3.7) in a form that better suits our purpose. If $x \in G$, let $t \in T$ be such that g is conjugate to t and let $H \in \mathfrak{t}$ be such that $\exp H = t$. The Riesz kernels are central functions; therefore,

$$|S_\varepsilon^{(n-1)/2}(x)| = |S_\varepsilon^{(n-1)/2}(\exp H)|.$$

Since the metric is invariant and the exponential mapping is a contraction we have

$$d(e, x) = d(e, t) \leq \|H\|.$$

The Weyl denominator D factors through the exponential mapping. This is most easily seen from the formula

$$D(H) = A(\rho)(H) = \sum_{w \in W} \det(w) \cdot \exp(2\pi i \rho(w \cdot H)) \qquad (H \in \mathfrak{t}).$$

Since G is simply connected, ρ is an integral form; therefore, $\exp H_1 = \exp H_2$ implies $D(H_1) = D(H_2)$. This allows us to define the central function $\tilde{D}$ on G whose restriction to T satisfies

$$\tilde{D}(\exp H) = D(H) \qquad (H \in \mathfrak{t}).$$

With x, t and H as before, we now have

$$\tilde{D}(x) = \tilde{D}(t) = \tilde{D}(\exp H) = D(H).$$

Note, that by the Weyl integral formula

$$\|\tilde{D}^{-1}\|_1 = (\operatorname{card} W)^{-1} \int_T |\tilde{D}(t)| \, d\omega(t) < \infty,$$

where ω is the normalized Haar measure on T.

With this in mind, estimate (3.7) reads

$$\sup_{\varepsilon > 0} |S_\varepsilon^{(n-1)/2}(x)| \leq A \cdot [(d(e,x))^{-n} + |\tilde{D}(x)|^{-1}] \qquad (x \in G).$$

Therefore,

$$|\mathscr{R}_\varepsilon^{(n-1)/2} b(x)| \leq A \cdot \left[\int_B |b(y)| (d(e, y^{-1}x))^{-n} \, d\sigma(y) + (|b| * |\tilde{D}|^{-1})(x)\right]$$

and we have

$$\begin{aligned} &\sigma(\{x \in G, \mathscr{R}^{(n-1)/2} b(x) > \lambda\}) \\ &\qquad \leq \sigma(B(e, 2\eta)) + \sigma(\{x \in G, \, d(e,x) \geq 2 \cdot \eta, \mathscr{R}^{(n-1)/2} b(x) > \lambda\}). \end{aligned}$$

Now, if $y \in B$ and $d(e,x) \geq 2 \cdot \eta$, then

$$d(e, y^{-1}x) = d(x,y) \geq d(e,x) - d(e,y) > \tfrac{1}{2} \cdot d(e,x).$$

Using this, $\|b\|_1 \leq 1$, $\lambda < \sigma(B) \leq C \cdot \eta^n$ and the fact that the convolution with the integrable function $|\tilde{D}^{-1}|$ is of strong type $(1,1)$ and therefore also of weak type $(1,1)$, we finally get

$$\begin{aligned}\sigma(\{x \in G, \mathscr{R}^{(n-1)/2}b(x) > \lambda\}) &\leq \sigma(B(e, 2\cdot\eta)) \\ &+ \sigma\left(\left\{x \in G, 2^n A[d(e,x)]^{-n} > \frac{\lambda}{2}\right\}\right) \\ &+ \sigma\left(\left\{x \in G, A(|b| * |\tilde{D}|^{-1})(x) > \frac{\lambda}{2}\right\}\right) \\ &\leq C \cdot (2\cdot\eta)^n + C \cdot 2^{n+1} \cdot A \cdot \lambda^{-1} + 2 \cdot A \cdot \|\tilde{D}^{-1}\|_1 \cdot \lambda^{-1} \leq \frac{C}{\lambda}.\end{aligned}$$

This finishes the proof. □

The space $B_p(G)$ generated by p-blocks is defined as usual; that is, we combine p-blocks using sequences (m_k) of coefficients satisfying $N((m_k)) < \infty$.

(3.10) THEOREM. *Let G be a compact simply connected and connected Lie group and $1 < p \leq \infty$. Then for every $f \in B_p(G)$ its Riesz means $\rho_\varepsilon^{(n-1)/2} f$ converge to f a.e. on G as $\varepsilon \to 0$.*

The proof is analogous to the proof of Theorem (2.15); we use Lemma (1.1) to add the weak type $(1,1)$ inequalities for p-blocks established in Theorem (3.9) and use (3.6) in a standard density argument to finish the proof.

REMARK. In the abelian case, that is, when the group is the torus T^n, the situation is different.

If $n = 1$ and $f \in B_\infty(T)$ then $\rho_\varepsilon^0 f$, which are (by agreement) just the partial sums of the Fourier series of f, converge to f a.e. on T. This was the motivating result for introducing spaces generated by blocks (see [**TW1**]). The proof is based on a simple estimate for the Dirichlet kernel and uses also the Carleson-Hunt theorem.

When $n > 1$, the Riesz kernel for the critical index

$$S_\varepsilon^{(n-1)/2}(x) = \sum_{m \in \mathbf{Z}^n} (1 - \varepsilon \cdot |m|^2)^{(n-1)/2} \cdot \exp(2\pi i m \cdot x)$$

is, as is well known, quite complicated (see [**SW**, VII, §4]). But, using a corresponding result for $\mathbf{R}^n$ and localization (both due to E. Stein) S. Lu, M. Taibleson and G. Weiss proved that for $f \in B_\infty(T^n) \cap L\log^+ L(T^n)$, $\rho_\varepsilon^{(n-1)/2} f$ converges to f a.e. (see [**LTW**]). It was observed by F. Soria, that the $L\log^+ L$ condition is necessary; the same function, constructed in [**SW**, VII, §4] to show that a.e. summability does not hold for $L^1(T^n)$ at the critical index is easily seen to be also an element of $B_\infty(T^n)$.

4. Constructions relevant to the theory of spaces generated by blocks on a space of homogeneous type. As mentioned in the Introduction, the spaces generated by blocks are closely connected with R. Fefferman's notion of entropy. The entropy of a set is a quantity which does not depend only on how large the set is measure-theoretically but also on the geometry of the set; that is, the entropy measures also how the set is spread out. The entropy of a function can be defined by using the entropy of its level sets. It turns out that the quasi-norm $N_q(f)$ of function $f \in B_q$ also depends on the size and the geometry of its level sets. The construction of functions f for which $N_q(f)$ is essentially larger than its L^1 norm is one of the basic results in the theory of spaces generated by blocks (see [**TW1**]).

In this section we make these constructions in the general setting of a nondiscrete space of homogeneous type.

Let X be a set. A quasi-distance on X is a nonnegative symmetric function d on $X \times X$ satisfying the quasi-triangle inequality

$$d(x,y) \le Q[d(x,z) + d(z,y)] \qquad (x,y,z \in X)$$

and such that $d(x,y) = 0$ iff $x = y$ $(x, y \in X)$. The sets $\{(x,y) \in X \times X, d(x,y) < 1/n\}$ form a base of a uniform structure on X and the collection of balls

$$B(x,r) = \{y \in X, d(x,y) < r\} \qquad (x \in X,\ r > 0)$$

is a base of neighbourhoods for the induced topology (caveat: the balls are not necessary open sets).

Let μ be a positive measure, defined on a σ-algebra of subsets of (X,d) which contains the open sets and the balls. The triple (X,d,μ) is a space of homogeneous type if μ satisfies the so-called doubling condition

$$0 < \mu(B(x,2r)) \le A \cdot \mu(B(x,r)) < \infty \qquad (x \in X,\ r > 0)$$

for some absolute constant A. Note that this doubling condition implies a slightly more general condition

$$0 < \mu(B(x,R)) \le A^{1+\log_2(R/r)} \cdot \mu(B(x,r)) < \infty$$

that allows us to compare measures on concentric balls with arbitrary radii. Let $1 < q \le \infty$. A q-block on (X,d,μ) is a measurable function b satisfying the localization condition

$$\operatorname{supp} b \subset B \quad \text{for some ball } B \subset X$$

and the size condition

$$\|b\|_q \le (\mu(B))^{(1-q)/q}.$$

If $\mu(X) < \infty$ then any function satisfying only the size condition with B replaced by X is also considered to be a q-block. $B_q(X,d,\mu)$ is the space of all functions f which can be written in the form $f = \sum m_j b_j$ where the b_j's are q-blocks and the numerical sequence (m_j) satisfies $N((m_j)) < \infty$. $B_q(X,d,\mu)$ is a quasi-Banach space (see the Introduction for more details on the quasi-norm).

When working with function spaces on spaces of homogeneous type it is very convenient to assume that the space is normal, that is, that the measure of a ball of radius r is of order r (see [**MS**] for the case of Lipschitz spaces). More precisely, (X,d,μ) is normal if there exist constants c_1 and c_2, $0 < c_1, c_2 < \infty$, such that

$$c_1 r \le \mu(B(x,r)) \le c_2 r \qquad (x \in X, \mu(\{x\}) < r < \mu(X)). \tag{4.1}$$

The sequence of lemmas that follow shows, that assuming normality of (X,d,μ) is not a restriction as far the spaces $B_q(X,d,\mu)$ are concerned.

(4.2) LEMMA [**MS**, THEOREM 2]. *Let d be a quasi-distance on X. There exists a quasi-distance d' on X such that the d'-balls are open sets in the d'-topology and*

$$C \cdot d(x,y) \leq d'(x,y) \leq \tilde{C} \cdot d(x,y) \qquad (x,y \in X)$$

for some constants $0 < C, \tilde{C} < \infty$, that is, d and d' are equivalent and therefore induce the same topology.

(4.3) LEMMA. *Let (X,d,μ) be a space of homogeneous type and let d' be a quasi-distance equivalent to d. Then (X,d',μ) is of homogeneous type, $B_q(X,d',\mu) = B_q(X,d,\mu)$ as sets and the corresponding quasi-norms $N_{q,d'}$ and $N_{q,d}$ are equivalent.*

PROOF. Clearly μ satisfies condition (4.1) with respect to d'. It is also easy to see that there exists a constant c such that, for every q-block b on (X,d,μ), $c \cdot b$ is a q-block on (X,d',μ). The rest of the proof is obvious. □

(4.4) LEMMA (SEE [**MS**, THEOREM 3]). *Let (X,d,μ) be a space of homogeneous type such that the d-balls are open sets. Then the function*

$$\delta(x,y) = \inf\{\mu(B), B \text{ a } d\text{-ball}, x,y \in B\} \quad \text{if } x \neq y,$$
$$\delta(x,x) = 0$$

is a quasi-distance on X, (X,δ,μ) is normal and the topologies induced on X by d and δ coincide.

REMARK. By definition of normality, δ-balls have the property

$$c_1 r \leq \mu(B_\delta(x,r)) \leq c_2 r \qquad (x \in X, \mu(\{x\}) < r < \mu(X))$$

for some finite positive constants c_1 and c_2. Let us say a few words on what happens for $r \notin (\mu\{x\}), \mu(X))$, $r > 0$. In a general space of homogeneous type $\mu(\{x\}) > 0$ if and only if x is an isolated point (see [**MS**, Theorem 1]). If δ is induced by d, as in the lemma above, then clearly

$$B_\delta(x,r) = \{x\} \qquad (x \in X,\ 0 < r \leq u(\{x\})).$$

If $\mu(X) < \infty$ and δ and d are as in the lemma then X is bounded; that is, $X = B_\delta(y,R)$ for some $y \in X$, $R > 0$. Indeed, we have $\delta(x,y) \leq \mu(X)$ for $x,y \in X$. Thus,

$$B_\delta(x,r) = X \qquad (x \in X,\ \mu(X) < r).$$

Finally, by left continuity of $\varphi(r) = \mu(B_\delta(x,r))$,

$$c_1 \mu(X) \leq \mu(B, \mu(X)) \leq \mu(X) \qquad (x \in X).$$

We do not know if $\mu(X) < \infty$ implies boundedness of X in general; this is also the reason for allowing exceptional q-blocks when $\mu(X) < \infty$.

(4.5) LEMMA. *Let (X,d,μ) and δ be as in the previous lemma. Then*

$$B_q(X,\delta,\mu) = B_q(X,d,\mu)$$

as sets and the corresponding quasi-norms are equivalent.

PROOF. Let b be a q-block on (X,d,μ); that is,

$$\operatorname{supp} b \subset B = B_d(x,r), \qquad \|b\|_q \leq (\mu(B))^{(1-q)/q}. \tag{4.6}$$

We want to show that for some constant c, independent of b, $c \cdot b$ is a q-block on (X, δ, μ).

If $y \in B$ then $\delta(x,y) \leq \mu(B)$; thus, $B \subset B_\delta(x,R)$ for any $R > \mu(B)$. Assume first that $\mu(B) < R < (1+\varepsilon)\mu(B) < \mu(X)$ for some $0 < \varepsilon < 1$. Using normality we have

$$\mu(B_\delta(x,R)) \leq c_2 R \leq c_2(1+\varepsilon)\mu(B) < 2c_2\mu(B).$$

Together with (4.6) and $\operatorname{supp} b \subset B_\delta(x,R)$ for our choice of R this shows that $c \cdot b$ is a q-block on (X,δ,μ) with c independent on b. On the other hand if $\mu(B) = \mu(X)$ then also $\mu(B_\delta(x,R)) = \mu(X)$. In this case we do not even have to multiply by a constant; b itself is a q-block on (X,δ,μ).

For the converse let b be a q-block with respect to δ, $\operatorname{supp} b \subset \tilde{B} = B_\delta(x,r)$, and $\|b\|_q \leq (\mu(\tilde{B}))^{(1-q)/q}$. If $y \in \tilde{B}$ then there exists a ball $B = B_d(z,\rho)$ such that $x, y \in B$ and $\mu(B) < r$. By the quasi-triangle inequality $d(x,y) < 2Q\rho$. We also have $B_d(x, 2Q\rho) \subset B_d(z, (2Q^2+Q)\rho)$. Using the doubling condition we get

$$\mu(B_d(x,2Q\rho)) \leq \mu(B_d(z,(2Q^2+Q)\rho)) \leq A^{\log_2(2Q^2+Q)+1}\mu(B) < C \cdot r.$$

Assume now that $\mu(X) = \infty$ and put

$$R = \operatorname{supp}\{s, \mu(B_d(x,s)) < C \cdot r\}.$$

Note that $R < \infty$ because $\lim \mu(B_d(x,s)) = \mu(X) = \infty$. Then, by the above, $\tilde{B} \subset B_d(x,R)$. Using left continuity of $\mu(B_d(x,\cdot))$ and normality

$$\mu(B_d(x,R)) \leq C \cdot r \leq C \cdot c_1^{-1}\mu(\tilde{B}) \quad \text{if } \mu(\{x\}) < r$$

and

$$\mu(B_d(x,R)) \leq C \cdot r \leq C \cdot \mu(\{x\}) \leq C \cdot \mu(\tilde{B}) \quad \text{if } r \leq \mu(\{x\}).$$

Therefore, $c \cdot b$ is a q-block on (X,d,μ), where c is independent of b.

If $\mu(X) < \infty$ the same argument works if $R < \infty$. So assume $R = \infty$. Then $\mu(X) \leq C \cdot r \leq C \cdot c_1^{-1}\mu(\tilde{B})$ if $\mu(\{x\}) < r < \mu(X)$, $\mu(X) \leq C \cdot r \leq C \cdot \mu(\tilde{B})$ if $r \leq \mu(\{x\})$, and $\mu(X) = \mu(\tilde{B})$ if $r \geq \mu(X)$. Thus, for an absolute constant c, $c \cdot b$ is an exceptional q-block with respect to the quasi-distance d.

To sum up, we have, up to a constant factor, the same q-blocks on (X,d,μ) and (X,δ,μ) and the lemma is proved. □

We now give the construction promised in the Introduction. Since it takes advantage of the gaps between points far apart we have to impose a mild restriction on X. Let us call a space of homogeneous type nondiscrete if it contains infinitely many nonisolated points.

(4.7) THEOREM. *Let (X,d,μ) be a nondiscrete space of homogeneous type, let K be a positive integer and $1 < q \leq \infty$. There exist functions $f \in B_q(X,d,\mu)$ such that f is a finite convex combination of blocks, $\|f\|_1 = 1$ and $N_q(f) = 1 + \log K$.*

PROOF. Lemmas (4.2), (4.3), (4.4) and (4.5) show that we may assume that our space X is normal.

We will construct disjoint balls B_k $(1 \leq k \leq K)$ in X such that the function

$$f = K^{-1} \cdot \sum_{k=1}^{K} (\mu(B_k))^{-1}\chi_{B_k} \tag{4.8}$$

will have the desired properties. Clearly $\|f\|_1 = 1$ and $N_q(f) \leq 1 + \log K$ because $(\mu(B_k))^{-1}\chi_{B_k}$ are q-blocks for every q, $1 < q \leq \infty$. The balls B_k will be chosen with large gaps between them and this will imply that (4.8) is the most efficient decomposition of f into q-blocks. We postpone the construction of the B_k's and deal with some preliminary reductions of the problem first.

Given an arbitrary decomposition

$$f = \sum m_j \cdot b_j \tag{4.9}$$

of f into q-blocks such that $N(m_j)) < \infty$ we have to show that

$$N((m_j)) \geq (1 + \log K). \tag{4.10}$$

Let $D = \bigcup B_k$. Since $f = \sum m_j(b_j \cdot \chi_D)$ and since $b_j \cdot \chi_D$ $(j \in \mathbb{N})$ are also q-blocks we may assume that $\operatorname{supp} b_j \subset D$.

The b_j's are q-blocks; therefore,

$$\operatorname{supp} b_j \subset D_j \quad \text{and} \quad \|b_j\|_q \leq (\mu(D_j))^{(1-q)/q} \qquad (j \in \mathbb{N}), \tag{4.11}$$

where the D_j are balls in X.

In the case when each D_j $(j \in \mathbb{N})$ is contained in one of the balls B_k the proof of (4.10) is easy. We just rearrange the terms in the absolutely converging series (4.9) as follows:

$$f = \sum_{k=1}^{K} \left(\sum_i m_i^{(k)} b_i^{(k)} \right) = \sum_{k=1}^{K} K^{-1} [\mu(b_k)]^{-1} \chi_{B_k}$$

where

$$\operatorname{supp} b_i^{(k)} \subset B_k \qquad (1 \leq k \leq K,\ i \in \mathbb{N}).$$

In what follows we will use the notation $\|m^{(k)}\|_1 = \sum |m_i^{(k)}|$. Since $\|b_i^{(k)}\|_1 \leq 1$ $(1 \leq k \leq K, i \in \mathbb{N})$ we have

$$\|m^{(k)}\|_1 \geq \sum_i |m_i^{(k)}|\, \|b_i^{(k)}\|_1 \geq \|K^{-1}[\mu(B_i)]^{-1}\chi_{B_i}\|_1 = K^{-1}$$

and, by the lattice property of $N(\cdot)$ (see [**TW1**, Lemma 5])

$$\begin{aligned} N((m_j)) &= N((m_1^{(1)}, m_2^{(1)}, \dots, m_1^{(K)}, m_2^{(K)}, \dots) \\ &\geq K^{-1} \cdot N\left[\left(\frac{(m_1^{(1)}}{\|m^{(1)}\|_1}, \frac{m_2^{(1)}}{\|m^{(1)}\|_1}, \dots, \frac{m_1^{(K)}}{\|m^{(K)}\|_1}, \frac{m_2^{(K)}}{\|m^{(K)}\|_1}, \dots \right)\right] \\ &= K^{-1} \sum_{k=1}^{K} \sum_i \frac{|m_i^{(k)}|}{\|m^{(k)}\|_1} \left(1 + \log \frac{K \cdot \|m^{(k)}\|_1}{|m_i^{(k)}|} \right) \\ &\geq 1 + \log K. \end{aligned}$$

It is in the general case, when the D_j's are allowed to intersect more than one B_k, that the gaps between the B_k's become important. We postpone the construction of the balls B_k once more; for now we need only to assume that they are disjoint.

We will show that by decomposing any b_j into the sum of its restrictions to the balls B_k we get a new decomposition of f which is more efficient than the old one.

The theorem will then be proved, because by cutting all b_j's like this, we reduce the general case to the special case established above.

Let us fix j, write $D = D_j$, $b = b_j$ and assume that D intersects more than one of the balls B_k. For $1 \le k \le K$ let

$$\begin{aligned} \tilde{m}_k &= (\mu(B_k))^{(q-1)/q} \|b \cdot \chi_{B_k}\|_q m_j, \\ a_k &= \frac{m_j}{\tilde{m}_k} b_k \cdot \chi_{B_k} \quad \text{if } B_k \cap D \neq \varnothing, \\ &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned} \tag{4.12}$$

Then the a_k's are q-blocks and

$$m_j b_j = \sum_{k=1}^{K} \tilde{m}_k a_k.$$

Therefore,

$$f = \sum_{k=1}^{K} \tilde{m}_k a_k + \sum_{i \neq j} m_i b_i.$$

To show that this decomposition of f is better than (4.9) we use the following simple observation about the quasi-norm N.

(4.13) LEMMA (SEE **[TW1]**). *Let $m = (m_i)$ be a numeric sequence and let*

$$\tilde{m} = (m_1, \ldots, m_{j-1}, \tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_K, m_{j+1}, \ldots),$$

where

$$\sum_{k=1}^{K} |\tilde{m}_k| \le \frac{|m_j|}{1 + \log K}.$$

Then $N(\tilde{m}) \le N(m)$.

To make use of the lemma we have to compare the sum of the $\tilde{m}_k$'s with m_j. Let

$$I = \{k \in \mathbb{N}, 1 \le k \le K, B_k \cap D \neq \varnothing\}$$

and recall that we are assuming card $I \ge 2$. Using (4.12), Hölder's inequality, $\operatorname{supp} b \subset \bigcup B_k$ and (4.11) we get

$$\begin{aligned} |m_j|^{-1} \sum_{k=1}^{K} |\tilde{m}_k| &= \sum_{k \in I} (\mu(B_k))^{(q-1)/q} \cdot \|b \cdot \chi_{B_k}\|_q \\ &\le \left(\sum_{k \in I} \mu(B_k) \right)^{(q-1)/q} \left(\sum_{k \in I} \|b \cdot \chi_{B_k}\|_q^q \right)^{1/q} \\ &= \left(\sum_{k \in I} \mu(B_k) \right)^{(q-1)/q} \cdot \|b\|_q \\ &\le \left[[\mu(D)]^{-1} \cdot \sum_{k \in I} \mu(B_k) \right]^{(q-1)/q}. \end{aligned}$$

Thus by Lemma (4.13) our goal is reached if

$$\mu(D) \geq (1+\log K)^{q/(q-1)} \cdot \sum_{k\in I} \mu(B_k) \qquad (\operatorname{card} I \geq 2). \tag{4.14}$$

We finally construct the balls B_k. Choose K distinct nonisolated points

$$x_1, x_2, \ldots, x_K \in X$$

and let $\delta = \min\{d(x_i, x_j), 1 \leq i, j \leq K, i \neq j\}$. Let $\gamma > 2Q$ be a fixed constant to be determined later. If $r \leq \delta/(Q+Q^2+\gamma Q^2)$ and $B_k = B(x_k, r)$ $(1 \leq k \leq K)$ then, as is easily seen,

$$d(y,z) > \gamma \cdot r \qquad (y \in B_i,\ z \in B_j,\ 1 \leq i,j \leq K,\ i \neq j).$$

By the quasi-triangle inequality we see that $\tilde{B}_k = B(x_k, \gamma r/2Q)$ $(1 \leq k \leq K)$ are disjoint. We still have flexibility in choosing r; that is, we can still make it smaller. In case $\mu(X) < \infty$ we make use of this possibility and require $0 < r < \gamma r/2Q < \mu(X)$. This enables us to use the normality of (X, d, μ) in all cases (note that the x_k are nonisolated points and thus $\mu(\{x_k\}) = 0$). We get

$$\mu(B_k) \leq c_2 r \leq \frac{2c_2 Q}{c_1 \gamma} \mu(\tilde{B}_k) \qquad (1 \leq k \leq K). \tag{4.15}$$

Let R be the radius and x the center of D. We assumed that $\operatorname{card} I \geq 2$, therefore

$$\gamma \cdot r < 2Q \cdot R. \tag{4.16}$$

Let us expand D to $\tilde{D} = B(x, (\gamma \cdot r)/2 + Q^2(R+r))$. If $k \in I$, that is, if B_k intersects D, using the quasi-triangle inequality, we see that $\tilde{B}_k \subset \tilde{D}$. Therefore,

$$\sum_{k\in I} \mu(\tilde{B}_k) \leq \mu(\tilde{D}). \tag{4.17}$$

By the doubling condition, (4.16) and since we assumed that $\gamma > 2Q$

$$\mu(\tilde{D}) \leq A^{(\log_2(\gamma r/2R + ((r+R)/R)\cdot Q^2)+1} \cdot \mu(D) \leq A^{\log_2(Q+3Q^2)+1} \cdot \mu(D).$$

This, together with (4.15) and (4.17), gives

$$\sum_{k\in I} \mu(B_k) \leq \frac{2c_2 \cdot Q}{c_1 \gamma} A^{\log_2(Q+3Q^2)+1} \cdot \mu(D)$$

and all that is left to do is to choose γ. Take

$$\gamma = \frac{2c_2 \cdot Q}{c_1 \gamma} A^{\log_2(Q+3Q^2)+1} \cdot (1+\log K)^{q/(q-1)}. \tag{4.18}$$

This establishes (4.14) and finishes the proof of the theorem. □

The theorem just proved can be used to derive many properties of the spaces B_q. In the case of the torus T, for example, M. Taibleson and G. Weiss showed that these spaces are not locally convex (see [**TW1**]). They also proved strict inclusions $B_q \subset B_p$, $B_q \neq B_p$, $1 < p < q < \infty$. For this latter result, however, one has to be able to build functions as described in Theorem (4.7) on small subsets of the space. This is possible on T because of the simple geometry; the fact needed for proving strict inclusions is then just a restatement of the basic construction. We show now, how this can be done in the general setting. We need a simple geometrical lemma.

(4.19) LEMMA. *Let (X, d, μ) be a normal space of homogeneous type without isolated points. Given a ball $B = B(x, R)$ and $K \in \mathbb{N}$, $K > 1$, there exist points $x_1, x_2, \ldots, x_K \in B$ such that*

$$d(x_i, x_j) \geq C \cdot K^{-1} \cdot \min\{\mu(X), R\} \qquad (i \neq j), \tag{4.20}$$

where C depends only on the constants of the space.

PROOF. Let $\rho = (c_1/c_2) \cdot K^{-1} \cdot \min\{\mu(X), R\}$. Let $x_1 = x$, $B_1 = B(x_1, \rho)$ and define B_k $(1 \leq k \leq K)$ by

$$B_k = B(x_k, \rho), \qquad x_k \in B \backslash (B_1 \cup B_2 \cdots \cup B_{k-1}).$$

This is possible because, by normality

$$\mu(B_1 \cup B_2 \cup \cdots \cup B_{k-1}) \leq \sum_{i=1}^{k-1} \mu(B_i) \leq (k-1)c_2\rho = \frac{k-1}{K} c_1 \min\{\mu(X), R\}$$
$$\leq \frac{k-1}{K} \cdot \mu(B) < \mu(B) \qquad (k \leq K),$$

so the set from which we pick x_k is nonempty. If $i \neq j$, $1 \leq i < j \leq K$, then $x_j \notin B_i$ which implies (4.20). □

(4.21) THEOREM. *Let (X, d, μ) be a normal space of homogeneous type without isolated points. Given $1 < q \leq \infty$, a ball B of radius R and $K \in \mathbb{N}$ there exists a subset $E \subset B$ such that*

$$\mu(E) \leq (1 + \log K)^{q/(1-q)} \cdot \mu(B) \tag{4.22}$$

and

$$N_q(\chi_E) \geq C \cdot (1 + \log K)^{1/(1-q)} \cdot \mu(B), \tag{4.23}$$

where C is a geometric constant.

PROOF. By Lemma (4.19) we choose $x_1, x_2, \ldots, x_K$ in the ball with same center as B and with radius $R/2Q$. Note that (4.20) is still valid with a smaller geometrical constant. Let, as in the proof of Theorem (4.7), $B_k = B(x_k, r)$ $(1 \leq k \leq K)$ where

$$\begin{aligned} r &= \min\left\{\frac{\delta}{Q + Q^2 + \gamma Q^2}, \frac{2Q\mu(X)}{\gamma}\right\}, \\ \delta &= \min\{d(x_i, x_j),\ 1 \leq i, j \leq K,\ i \neq j\}, \\ \gamma &= C \cdot (1 + \log K)^{q(1-q)} \end{aligned} \tag{4.24}$$

and C is a geometrical constant (see (4.18)). By the proof of Theorem (4.7) (see (4.14))

$$\sum_{k=1}^{K} \mu(B_k) \leq (1 + \log K)^{q/(1-q)} \cdot \mu(D)$$

for every ball D that intersects $B_1, B_2, \ldots, B_K$. Our ball B does that and this gives (4.22) for $E = \bigcup B_k$.

It is easy to see that $E \subset B$; we have chosen the centers x_k of the balls B_k close enough to the center of B. By the theorem we also have

$$N_q\left(\sum_{k=1}^{K} [\mu(B_k)]^{-1} \cdot \chi_{B_k}\right) = K \cdot (1 + \log K). \tag{4.25}$$

We now establish (4.23) as follows. Let $S = \min\{\mu(X), R\}$. By (4.20) $\delta \geq C \cdot S \cdot K^{-1}$ and, therefore,

$$r \geq C \cdot S \cdot K^{-1} \cdot \gamma^{-1}.$$

Using normality (note that $r < \mu(X)$), the estimate above and the fact that $S \geq C \cdot \mu(B)$ (if $R < \mu(X)$ then $S = R \geq C \cdot \mu(B)$ by normality, if $R \geq \mu(X)$ then $S = \mu(X) \geq \mu(B)$) we get

$$\mu(B_k) \geq c_1 r \geq C \cdot S \cdot K^{-1}\gamma^{-1} \geq C \cdot K^{-1}\gamma^{-1}\mu(B).$$

Therefore

$$\chi_E \geq C \cdot K^{-1}\gamma^{-1}\mu(B) \sum_{k=1}^{K} [\mu(B_k)]^{-1}\chi_{B_k}$$

and since N_q satisfies the lattice property (that is, if $|f| \leq |g|$ then $N_q(f) \leq N_q(g)$) we get, using (4.25) and (4.24) the desired inequality

$$\begin{aligned} N_q(\chi_E) &\geq C \cdot K^{-1}\gamma^{-1}\mu(B) \cdot N_q\left(\sum_{k=1}^{K} [\mu(B_k)]^{-1}\chi_{B_k}\right) \\ &= C \cdot \gamma^{-1}\mu(B) \cdot (1 + \log K) = C \cdot (1 + \log K)^{1/(1-q)}. \quad \square \end{aligned}$$

The theorem can be used to prove strict inclusion relations among the spaces $B_q(X, d, \mu)$ exactly as in [**TW1**] for the case of the torus T.

5. On summation of weak type (1, 1) inequalities. Lemma (1.1) on summing of weak type (1, 1) inequalities is one of the cornerstones of the theory of spaces generated by blocks. It is a measure-theoretic result and can be compared to Beppo Levi's theorem.

Let (X, μ) be a measure space and f a measurable function on X. The weak L^1 quasi-norm of f is

$$\|f\|_{1,\infty} = \sup_{\lambda > 0} \lambda \cdot \mu(\{x \in X, |f(x)| > \lambda\})$$

and we denote by $L^{1,\infty}(X)$ the space of measurable functions f on X with $\|f\|_{1,\infty} < \infty$. Lemma (1.1) says that for a sequence (f_k) of measurable functions

$$\left\|\sum_k |f_k|\right\|_{1,\infty} \leq 4 \cdot \sum_k \|f_k\|_{1,\infty} \cdot \left(1 + \log \frac{\sum \|f_j\|_{1,\infty}}{\|f_k\|_{1,\infty}}\right),$$

that is, if

(5.1) $$N((\|f_k\|_{1,\infty}) < \infty$$

then

(5.2) $$\sum |f_k| \in L^{1,\infty}(X)$$

and therefore

(5.3) $\sum f_k$ converges absolutely a.e. on X.

In Beppo Levi's theorem for $L^1(X)$ the hypothesis $\sum \|f_k\|_1 < \infty$ is not only sufficient but also necessary for the conclusion $\sum |f_k| \in L^1(X)$. In this subsection we address the question of necessity of condition (5.1) for (5.2) and show that in a sense (5.1) is best possible.

The following simple examples show that (5.1) is a strong condition and is far from being necessary for (5.2).

(a) Let (c_k) be a sequence such that $\sum |c_k| < \infty$ and $N((c_k)) = \infty$. Then the series $\sum f_k(x) = \sum c_k \cdot x^{-1}$ converges a.e. on $\mathbf{R}$ and defines a function in $L^{1,\infty}(\mathbf{R})$. But $\|f_k\|_{1,\infty} = 2 \cdot |c_k|$ and so (5.1) fails.

(b) Let (c_k) be as in (a) and let $I = [0,1]$. With each $x \in I$ we associate its dyadic expansion and let $R_k(x)$ be the kth digit in this expansion; we require that the expansion of x contains infinitely many zeroes. Then $\sum c_k \cdot R_k \in L^1(I) \subset L^{1,\infty}(I)$ but $\|c_k \cdot R_k\|_{1,\infty} = \frac{1}{2}|c_k|$ so (5.1) fails again.

Note that the functions f_k in Example (a) are multiples of the typical nonintegrable $L^{1,\infty}$ function x^{-1} and that the functions in Example (b) are independent as random variables on the probability space I. However, we will show that if we have both independence and x^{-1}-like behaviour for a sequence (f_k) on a probability space then (5.1) is necessary for (5.3). For such sequences (5.1), (5.2) and (5.3) are therefore equivalent.

(5.4) PROPOSITION. *Let (X,μ) be a finite measure space, $\mu(X) = 1$, and let (f_k) be a sequence of functions in $L^{1,\infty}(X)$ satisfying*

(i) *the f_k are independent as random variables on the probability space (X,μ), and*

(ii) *for every k and every $\lambda > 0$*

$$\mu(\{x \in X, |f_k(x)| > \lambda\}) = \min(1, \|f_k\|_{1,\infty} \cdot \lambda^{-1}) \tag{5.5}$$

(in particular $|f_k(x)| > \|f_k\|_{1,\infty}$ a.e.). If the series $\sum |f_k|$ converges a.e. on X then $N((\|f_k\|_{1,\infty}))$ is finite and therefore $\sum f_k \in L^{1,\infty}(X)$.

PROOF. We may assume that the f_k are nonnegative because if (f_k) satisfies the hypothesis, so does $(|f_k|)$ and if the conclusion is valid for $(|f_k|)$ it is also valid for (f_k).

The proof is an application of Kolmogorov's three series criterion for a.e. convergence of a series of independent random variables in terms of their truncations. The criterion, stated in measure-theoretic terminology says (see [**H**, p. 199]):

Let (f_k) be a sequence of independent random variables on (X,μ) and define the truncation of f_k at height $c > 0$ by

$$\begin{aligned} f_k^c(x) &= f_k(x) \quad \text{if } |f_k(x)| \le c, \\ &= 0 \qquad\quad \text{otherwise.} \end{aligned}$$

Then the series $\sum f_k$ converges a.e. on X if and only if for some $c > 0$ the following three series converge:

$$\sum \mu(\{x \in X, |f_k(x)| > c\}),$$

$$\sum \int_X f_k^c(x)\, d\mu(x),$$

$$\sum \int_X \left[f_k^c(x) - \int_X f_k^c(t)\, d\mu(t) \right]^2 d\mu(x).$$

If the series converge for some $c > 0$ then they do so for every $c > 0$.

We now have for any fixed $c > 0$

$$\begin{aligned} c \cdot \mu(\{x \in X, |f_k(x)| > c\}) &+ \int_X f_k^c(x)\, d\mu(x) \\ &= \int_X \min(c, f_k(x))\, d\mu(x) \\ &= \int_0^c \mu(\{x \in X, f_k(x) > \lambda\})\, d\lambda = \int_0^c \min(1, \|f_k\|_{1,\infty} \cdot \lambda^{-1})\, d\lambda. \end{aligned}$$

The last integral equals $\|f_k\|_{1,\infty}(1 + \log c \cdot \|f_k\|_{1,\infty}^{-1})$ if $\|f_k\|_{1,\infty} \le c$ and c otherwise, that is, it equals

$$\min(c, \|f_k\|_{1,\infty}) \cdot (1 + \log^+ c \cdot \|f_k\|_{1,\infty}^{-1}).$$

Now, if the series $\sum f_k$ is convergent a.e. on X then the series

$$c \cdot \sum \mu(\{x \in X, |f_k(x)| > c\})$$

and

$$\sum \int_X f_k^c(x)\, d\mu(x)$$

converge for any $c > 0$ by Kolmogorov's criterion. Thus for any $c > 0$

$$\sum \min(c, \|f_k\|_{1,\infty}) \cdot (1 + \log^+ c \cdot \|f_k\|_{1,\infty}^{-1}) < \infty.$$

This series is clearly equiconvergent with $N((\|f_k\|_{1,\infty}))$ and the proposition is proved. □

REMARKS. Note that it is enough to assume (5.5) for $\lambda < C$, where C is a constant independent of k, that is, it is enough to require x^{-1}-like behaviour of f_k's on the set where they take on small values.

Let us give an example of a sequence (f_k) on $I = [0,1]$ satisfying (i) and(ii) of (5.4). Let

$$S_k(x) = \sum_{j=1}^{\infty} 2^{-j} \cdot R_{\varphi(k,j)}(x)$$

where $\varphi \colon \mathbf{N}^2 \to \mathbf{N}$ is a bijection and the R_n are functions of Example (b). It is easy to see that the S_k are independent and uniformly distributed over $[0,1]$. Finally let (c_k) be a numeric sequence and let $f_k = c_k \cdot (1 - S_k)^{-1}$. We have $\|f_k\|_{1,\infty} = |c_k|$, (i) and (ii) are valid, so by the proposition $\sum f_k$ converges a.e. in I if and only if $N((c_k)) < \infty$.

We saw in Example (b) that independence and a.e. convergence do not necessarily imply (5.1) for sequences in $L^{1,\infty}$. An interesting related question, to which we do not know the answer, is whether they imply (5.2) in absence of x^{-1}-like behaviour of the terms of the sequence.

BIBLIOGRAPHY

[**B1**] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. **40** (1936), 175–207.

[**B2**] ——, *Summation of derived Fourier series*, Ann. of Math. **37** (1936), 345–356.

[**BC**] A. Bonami and J. L. Clerc, *Sommes de Cesàro et multiplicateurs des developpements en harmoniques spheriques*, Trans. Amer. Math. Soc. **183** (1973), 223–263.

[**BD**] T. Broecker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Math., vol. 98, Springer-Verlag, 1985.

[**C**] J. L. Clerc, *Sommes de Riesz et multiplicateurs sur un groupe de Lie compact*, Ann. Inst. Fourier (Grenoble) **24** (1974), 149–172.

[**CTW**] L. Colzani, M. H. Taibleson and G. Weiss, *Maximal estimates for Cesàro and Riesz means on spheres*, Indiana Univ. Math. J. **33** (1984), 873–889.

[**F**] R. Fefferman, *A theory of entropy in Fourier analysis*, Adv. in Math. **30** (1978), 171–201.

[**H**] P. R. Halmos, *Measure theory*, Van Nostrand, 1950.

[**LTW**] S. Z. Lu, M. H. Taibleson and G. Weiss, *On the almost everywhere convergence of Bochner-Riesz means of multiple Fourier series*, Harmonic Analysis Proc., Minneapolis, 1981, Lecture Notes in Math., vol. 908, Springer-Verlag, 1982.

[**MS**] R. A. Macias and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. **33** (1979), 257–270.

[**MTW**] Y. Meyer, M. H. Taibleson and G. Weiss, *Some functional analytic properties of the spaces B_q generated by blocks*, Indiana Univ. Math. J. **34** (1985), 493–516.

[**So**] F. Soria, *Characterizations of classes of functions generated by blocks and associated Hardy spaces*, Indiana Univ. Math. J. **34** (1985), 463–492.

[**STW**] E. M. Stein, M. H. Taibleson and G. Weiss, *Weak type estimates for maximal operators on certain H^p classes*, Suppl. Rend. Circ. Mat. Palermo **1** (1981), 81–97.

[**SW**] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, 1971.

[**T**] M. H. Taibleson, *Estimates for finite expansions of Gegenbauer and Jacobi polynomials*, Recent Progress in Fourier Analysis (I. Peral and J.-L. Rubio de Francia, eds.), North-Holland Math. Stud., vol. 111, North-Holland, 1985.

[**TW1**] M. H. Taibleson and G. Weiss, *Certain function spaces connected with almost everywhere convergence of Fourier series*, Proc. Conf. on Harmonic Analysis in honor of Antoni Zygmund, vol. I, Wadsworth, 1982.

[**TW2**] ——, *Spaces generated by blocks*, Probability Theory and Harmonic Analysis (J.-A. Chao and W. A. Woyczinski, eds.), Dekker, 1986.

INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, E. K. UNIVERSITY OF LJUBLJANA, LJUBLJANA, YUGOSLAVIA